Notes:

- This tutorial on the Bloch sphere deals only with two-state systems, such as a single qubit. Additionally, the tutorial deals only with pure states, and not mixed states.
- When a qubit is measured in a particular basis, it yields a particular eigenvalue corresponding to specific basis state to which the state collapsed. Thus, because of the state collapse, we know the eigenvalue and the state after the measurement. In this tutorial, the term "measurement outcome" will generally refer to the collapsed state after the measurement.

Bloch sphere

Warm-up: Construction

You should be able to:

- *Describe the connection between a two-dimensional quantum state written in Dirac notation and its representation on the Bloch sphere*
- *Identify where the states $|0\rangle, |1\rangle, |\pm\rangle, |\pm i\rangle$ are on the Bloch sphere*
- *Identify that multiplying by an overall phase does not change the state, but modifying a relative phase between the basis states does*

Reading the Bloch sphere:

In the diagrams used in this tutorial, the sphere is oriented so that both the x- and y-axes appear offset from the center by equal amounts. For practical reasons, the positive z-axis is also slightly tilted toward the viewer. A beach ball is illustrated below to get the point across.

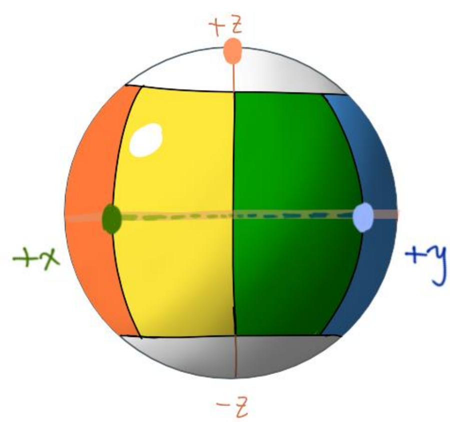


Figure 1: With the z-axis placed straight up and down (vertically), the x-y plane, viewed edge-on, is completely flat and nearly invisible!

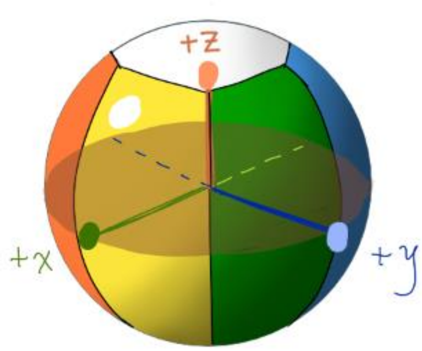


Figure 2: The top half of the ball is angled slightly toward the viewer. (Notice that the z-, x-, and y-axes do reach the edge of the ball, even though it does not look that way in this perspective.) This is the view from which all the following Bloch sphere diagrams are drawn.

Orientation

What is the Bloch sphere?

The Bloch sphere is a way of visually representing states of a two-state system, such as a qubit. When the given state and the measurement basis are plotted on the Bloch sphere, one can relatively easily get a sense of the outcomes of a measurement and their probabilities.

The Bloch sphere is a spherical surface of radius 1, with polar angle θ and azimuthal angle ϕ that are the same as the angles used in spherical coordinates (as conventionally defined in physics). That is, θ begins from the positive z-axis and sweeps toward the equator. Meanwhile, when looking down the positive z-axis, ϕ begins from the positive x-axis and sweeps counterclockwise about the z-axis. Any point on the surface of the Bloch sphere represents a valid pure quantum state.

I've chosen a point on the Bloch sphere… what does it mean?

Any point on the Bloch sphere can be specified with a unique θ and ϕ. Each state $|q\rangle$ that can be plotted on the Bloch sphere is represented in the $\{|0\rangle, |1\rangle\}$ basis as $|q\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$. (When plotted in two-dimensional Hilbert space with complex coefficients, the angle that the state forms with the $|0\rangle$ axis is $\frac{\theta}{2}$.) We have the freedom to choose an overall phase such that the $|0\rangle$ component is real and non-negative, and we usually restrict the angles such that $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$, to ensure a one-to-one mapping.

What does an orthonormal basis look like on the Bloch sphere?

For any qubit, each measurement has a unique basis associated with it, which includes two orthonormal basis states, one for each possible outcome. To obtain an orthonormal basis on the Bloch sphere, simply locate the two ends of any possible diameter of the sphere. For instance, if the $|0\rangle$ state points straight up, then the $|1\rangle$ state points straight down. Likewise, states that point directly left and right (or any two states on "opposite" points of the Bloch sphere) are also orthonormal to each other and can be used as measurement basis states. What's more, because of the complete spherical symmetry, we are free to label any pair of such basis states as the standard basis $\{|0\rangle, |1\rangle\}$, and thus define the z-axis in any direction we would like to choose.

1. There is no number 1. (Hooray!)

2. Verify that a complex number in the form $e^{i\phi} = \cos\phi + i\sin\phi$ has unit modulus (i.e., magnitude of 1) by calculating $\left|e^{i\phi}\right|^2$. Is the state $\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$ normalized for all values of θ and ϕ?

3. Consider four qubits in the following states:
   - $|q_{A1}\rangle = a|0\rangle + b|1\rangle$
   - $|q_{A2}\rangle = -a|0\rangle - b|1\rangle$
   - $|q_{B1}\rangle = a|0\rangle - b|1\rangle$
   - $|q_{B2}\rangle = e^{i\phi}a|0\rangle - e^{i\phi}b|1\rangle$

(A) List the outcomes, and calculate and compare the probabilities of measuring those outcomes, when the states $|q_{A1}\rangle$ and $|q_{A2}\rangle$ are measured in the $\{|0\rangle, |1\rangle\}$ basis.

(B) List the outcomes, and calculate and compare the probabilities of measuring these outcomes, when the states $|q_{B1}\rangle$ and $|q_{B2}\rangle$ are measured in the $\{|0\rangle, |1\rangle\}$ basis.

(C) Is the probability of measuring $|0\rangle$ or $|1\rangle$ any different between parts (A) and (B)?

(D) The $\{|+\rangle, |-\rangle\}$ basis consists of the states $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ when expressed in the $\{|0\rangle, |1\rangle\}$ basis.

List the outcomes, and calculate and compare the probabilities of measuring those outcomes, when the states $|q_{A1}\rangle$ and $|q_{A2}\rangle$ are measured in the $\{|+\rangle, |-\rangle\}$ basis.

Hint: $|q\rangle = a|0\rangle + b|1\rangle$ is expressed in the $\{|+\rangle, |-\rangle\}$ basis as $|q\rangle = \frac{a+b}{\sqrt{2}}|+\rangle + \frac{a-b}{\sqrt{2}}|-\rangle$.

(E) List the outcomes, and calculate and compare the probabilities of measuring these outcomes, when the states $|q_{B1}\rangle$ and $|q_{B2}\rangle$ are measured in the $\{|+\rangle, |-\rangle\}$ basis.

(F) Is the probability of measuring $|+\rangle$ or $|-\rangle$ any different between parts (D) and (E)?

(G) Is it possible to tell the states $|q_{A1}\rangle$, $|q_{A2}\rangle$, $|q_{B1}\rangle$, or $|q_{B2}\rangle$ apart from one another in the $\{|0\rangle, |1\rangle\}$ basis? In the $\{|+\rangle, |-\rangle\}$ basis?

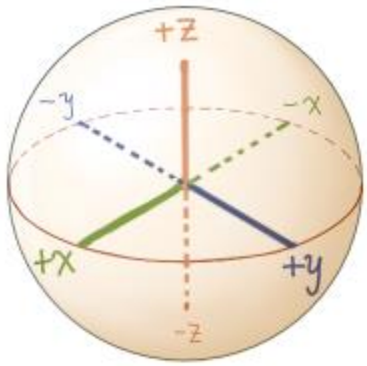
Checkpoint
• For the state of a qubit (e.g., $\|q\rangle = a\|0\rangle + b\|1\rangle$ in the $\{\|0\rangle, \|1\rangle\}$ basis), multiplying a and b by the same phase $e^{i\phi}$ (an "overall phase") $\|q\rangle$ represents an equivalent state. In contrast, multiplying only one of a or b by a phase (thus changing their "relative phase") will change the state. • When states differ by a relative phase in a particular basis, measurement in that basis is unable to distinguish between them. However, transforming to any other basis will allow one to tell such states apart. • Note that $\|q_{A2}\rangle = -a\|0\rangle - b\|1\rangle$ is a special case of $e^{i\phi}a\|0\rangle + e^{i\phi}b\|1\rangle$ where $e^{i\phi} = -1$ and $\phi = \pi$. Other simple values of $e^{i\phi}$ include $e^{i\frac{\pi}{2}} = i$ and $e^{i\frac{3\pi}{2}} = -i$.

4. A state $|q\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$ on the Bloch sphere (which is a unit sphere) can be visualized as a unit vector $\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$ in spherical coordinate space, with Cartesian coordinates $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$, and $z = \cos\theta$. (Notice that here, θ is no longer divided by 2; recall that $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. We will explain later why θ goes to $\frac{\theta}{2}$, while ϕ remains unchanged.)

   Find the spherical coordinate vectors (known in the context of the Bloch sphere as the "Bloch vectors") that correspond to the following states. Fill out the table to make the process easier.

State	Value of θ?	Value of ϕ?	$\hat{\mathbf{r}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$
$\|0\rangle = 1\|0\rangle + 0\|1\rangle$			
$\|1\rangle$			
$\|+\rangle = \frac{1}{\sqrt{2}}\|0\rangle + \frac{1}{\sqrt{2}}\|1\rangle$			
$\|-\rangle = \frac{1}{\sqrt{2}}\|0\rangle - \frac{1}{\sqrt{2}}\|1\rangle$			
$\|+i\rangle = \frac{1}{\sqrt{2}}\|0\rangle + \frac{i}{\sqrt{2}}\|1\rangle$			
$\|-i\rangle = \frac{1}{\sqrt{2}}\|0\rangle - \frac{i}{\sqrt{2}}\|1\rangle$			

Draw and label each of the above states on the Bloch sphere below. Treat the Bloch vector as a spherical coordinate vector with radius 1, polar angle θ defined with respect to from the positive *z*-axis, and azimuthal angle ϕ defined with respect to from the positive *x*-axis.



Check your answers with the simulation on the following site: https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/blochsphere/blochsphere.html

For each state in the left-hand column, insert your values into the simulation for θ and ϕ and verify that the Bloch vector is where you said it was.

5. Notice that a state written in the form $\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$ has a real and non-negative $|0\rangle$ component. If the $|0\rangle$ component of a state is negative, complex or imaginary, we can still plot it on the Bloch sphere, but we must first multiply it by an overall phase (a factor of $e^{i\psi}$ for some angle ψ) to find the values of θ and ϕ.

   For each of the following states, multiply by an overall phase $e^{i\psi}$ that makes the $|0\rangle$ component real and positive. Write the resulting state. (Note: you must explicitly find $e^{i\psi}$, but you do not need to find ψ itself.)

   (A) $|a\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

   (B) $|b\rangle = \frac{i}{\sqrt{3}}|0\rangle + \frac{1+i}{\sqrt{3}}|1\rangle$

6. Starting from the state $|q\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{1-i}{\sqrt{3}}|1\rangle$, find the values of θ and ϕ that express $|q\rangle$ in the form $\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$. Again, $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$.

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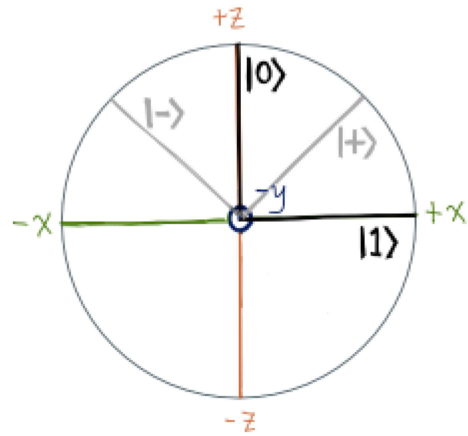
Hint: Use the amplitude of $|0\rangle$ to find the value of $\frac{\theta}{2}$ first. Then, using this angle, factor out $\sin\frac{\theta}{2}$ from the complex amplitude of $|1\rangle$ and express the remaining complex number in the form $e^{i\phi} = \cos\phi + i\sin\phi$ (i.e., find the angle ϕ). You are guaranteed to find a unique answer because the states are normalized.

Note that in these expressions, θ is divided by 2! The angle made in Hilbert space (in which coefficients are generally complex) between the state and the $|0\rangle$ axis is exactly half of the angle θ made on the Bloch sphere between the corresponding Bloch vector and the $|0\rangle$ axis.

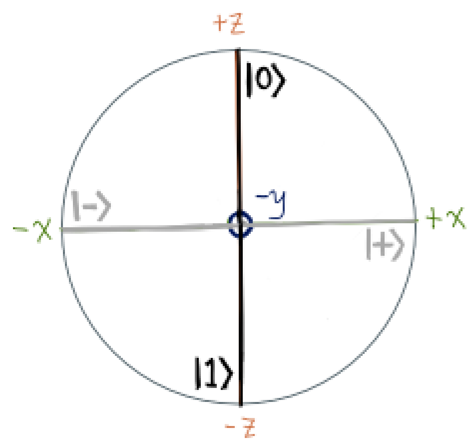
Checkpoint
• If you are having trouble reasoning whether you should multiply or divide by the factor of 2, a simple way to remember is that on the Bloch sphere, angles are always *bigger*, while the same angle in Hilbert space is *smaller*. (Remember: Bloch is bigger.) • When $\theta = 0$ or π, ϕ can actually take any value. Does this make sense?

Consider the following conversation between two students.

Student 1: Orthonormal basis states, like $|0\rangle$ and $|1\rangle$, are oriented at right angles (90°) from each other on the Bloch sphere, like in the diagram below. The same is true for $|+\rangle$ and $|-\rangle$.



Student 2: No, I thought orthonormal basis states were found opposite each other on the Bloch sphere. So if $|0\rangle$ is placed on the north pole, then $|1\rangle$ would be on the south pole. $|+\rangle$ and $|-\rangle$ would also be on opposite poles, as seen below.



Which student(s), if any, do you agree with? Explain.

Checkpoint
• On the Bloch sphere, orthonormal basis states are found exactly opposite each other, or equivalently, they are spaced 180° apart. They are *not* found 90° apart.

Consider the following conversation between several students.

Student 1: The x-, y-, and z-axes are orthogonal to each other on the Bloch sphere, which is a sphere with unit radius. The states $|0\rangle$ and $|+\rangle$ have unit length and they lie at 90° from each other, so they form an orthonormal basis.

Student 2: We can express any point on the Bloch sphere using spherical coordinates r, θ, and ϕ, which can be converted to Cartesian coordinates x, y, and z. I wonder what a state would look like if we decided to express it in the $\{|0\rangle, |+\rangle\}$ basis, which would be like making a measurement in the *x*-*z* plane. You would take projective measurements along either the $|0\rangle$ state or the $|+\rangle$ state.

Student 3: Slow down. I thought an orthonormal basis for a qubit meant that taking the inner product of a basis state with itself like $\langle 0|0\rangle$ gives you 1, and taking the inner product with the other basis state like $\langle 0|1\rangle$ would give you 0. But if you take the inner product of $|0\rangle$ with $|+\rangle$, you don't get 0 *or* 1!

Student 4: Right. It isn't very useful to express the states of a qubit in a "basis" consisting of $|0\rangle$ and $|+\rangle$. You could do it, but these states don't form an orthonormal basis, and they would be much more difficult to work with as a result. This would be similar to choosing your *x*- and *y*-axes in introductory physics to be at, for example, 45° instead of 90° apart.

Student 3: It also doesn't make sense to make a measurement in the *x*-*z* plane.

Student 4: Right, the Bloch sphere itself doesn't show amplitudes associated with projective measurements in an intuitive way, because θ is twice as big and needs to be divided by 2.

Which student(s), if any, do you agree with? Explain.

Checkpoint
• Pairs of states found 90° apart from each other on the Bloch sphere, such as $\|0\rangle$, $\|+\rangle$, and $\|+i\rangle$ (which correspond to the $+z$, $+x$, and $+y$-axes respectively), do ***not*** comprise an orthonormal basis. Pairs of such states could still be used as a basis, but the basis would not be orthonormal. • Put another way, the Bloch sphere is situated in a 3-D space which is ***not*** Hilbert space.

Consider the following conversation between several students.

Student 1: But why would a basis consisting of $|0\rangle$ and $|+\rangle$ not be considered an orthonormal basis, since the states are orthogonal and normalized?

Student 2: They're not actually orthogonal, are they? If they were, then $\langle 0|+\rangle = 0$ would be true, but $\langle 0|+\rangle$ actually equals $\frac{1}{\sqrt{2}}$.

Student 3: Right. You shouldn't get the "orthogonal" basis states of the qubit confused with the 90-degree angles on the Bloch sphere representation that correspond to the x-, y-, and z-axes. Remember, orthonormal basis states are actually found on opposite ends of the Bloch sphere, because the angle θ between them appears twice as big on the Bloch sphere. $|0\rangle$ and $|+\rangle$, which are $\Delta\theta = 90°$ apart on the Bloch sphere, actually have an angle $\frac{\Delta\theta}{2} = 45°$ between them when viewed in Hilbert space.

Student 1: Oh, I see! So for any state on the Bloch sphere, to find the state that is orthonormal to it, you can simply go to the point directly on the other side of the sphere, i.e., diametrically opposite, and you are guaranteed to have orthonormal basis states!

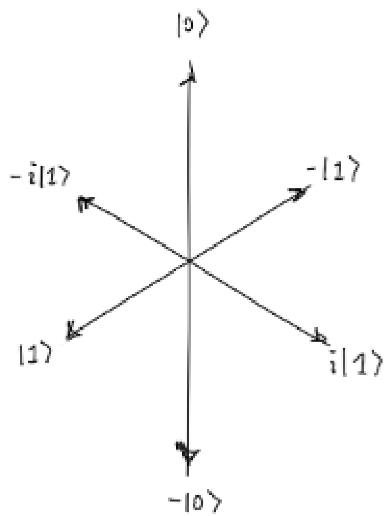
Student 2: And this way, each state will only ever have one other state on the Bloch sphere that's orthonormal to it, so the orthonormal basis will be unique!

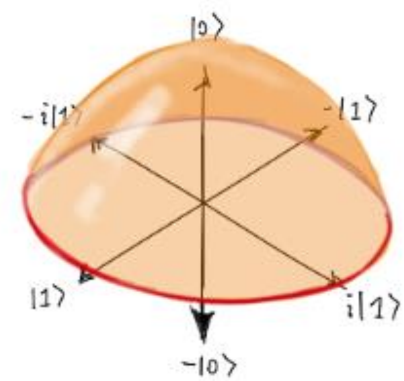
Which student(s), if any, do you agree with? Explain.

Checkpoints (collected)
• If you are having trouble reasoning whether you should multiply or divide by the factor of 2, a simple way to remember is that on the Bloch sphere, angles are always *bigger*, while the same angle in Hilbert space is *smaller*. (Remember: Bloch is bigger.) • When $\theta = 0$ or π, ϕ can actually take any value. • On the Bloch sphere, orthonormal basis states are found exactly opposite each other, or equivalently, they are spaced $\Delta 180°$ apart. They are *not* found 90° apart. • Pairs of states found 90° apart from each other on the Bloch sphere, such as $\|0\rangle$, $\|+\rangle$, and $\|+i\rangle$ (which correspond to the $+z$, $+x$, and $+y$-axes respectively), do ***not*** comprise an orthonormal basis. Pairs of such states could still be used as a basis, but the basis would not be orthonormal. The Bloch sphere is situated in a 3-D space which is ***not*** Hilbert space.

How do you go from Hilbert space to the Bloch sphere?

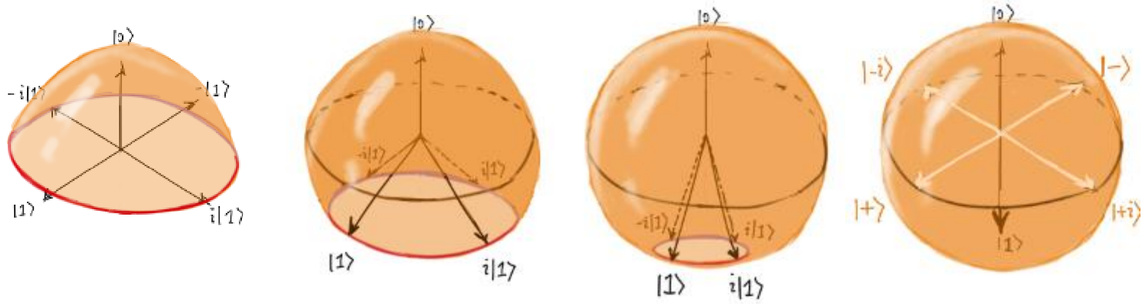
First we set the positive and negative $|0\rangle$ states on the *z*-axis, and we can set a complex plane of all possible complex components of $|1\rangle$ on the *x*-and *y*-axes. If we choose the overall phase so that we only consider states with a real and positive $|0\rangle$ component, then the $|0\rangle$ component needs neither its own complex plane nor the entire negative half. Since all states have unit length, the set of all possible states then forms a hemispherical surface as shown below.





Because the four states $|1\rangle, -|1\rangle, i|1\rangle$, and $-i|1\rangle$ differ by only an overall phase, they all represent the same physical state. In fact, this is true of *every* state on the circular edge of the hemispherical surface (shown in red), and these are also the *only* states on the hemisphere that differ only by an overall phase rather than a relative phase. This is because the value of the $|0\rangle$ component is zero at the equator.

Imagine fashioning this hemisphere out of some elastic, rubbery material. We then simply stretch that circular edge along a full unit sphere to meet at the south pole (the negative *z*-axis). We have effectively turned a half sphere into a full sphere by doubling the apparent value of θ everywhere, so to recover the original state, we divide the new θ by 2.

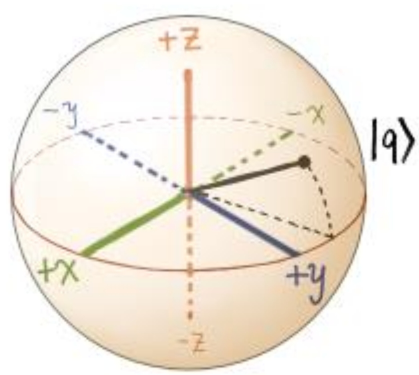


The result is the Bloch sphere. Note that there are no states on the surface that differ by an overall phase (since the unit circle of such states has now been merged into a single point). Any state in 2-D Hilbert space can be multiplied by an overall phase to yield a point on the Bloch sphere.

7. How many possible pure quantum states can be represented on the Bloch sphere?

   Remember that a pure state is represented on the surface of a Bloch sphere by a point, and that each state on the Bloch sphere has a corresponding (but not necessarily unique) state in two-dimensional Hilbert space.

8. Consider a qubit (illustrated below, along with the angles that it makes with respect to the *x*-axis and the *x*-*y* plane) in the state $|q\rangle = a|0\rangle + b|1\rangle$, with $a = 0.891$ and $b = -0.117 + 0.438i$.



(A) When measured in the $\{|0\rangle, |1\rangle\}$ basis, what is the probability that $|q\rangle$ will collapse into the state $|0\rangle$? (Use a calculator if you can; the one on any smartphone should suffice.)

(B) When measured in the $\{|0\rangle, |1\rangle\}$ basis, what is the probability that $|q\rangle$ will collapse into the state $|1\rangle$?

Check your answers above with the simulation again. Use $\theta = \frac{3\pi}{10}$ and $\phi = \frac{7\pi}{12}$ to represent the state $|q\rangle$. Click on the "Show theoretical measurement outcome probabilities" box to see them.
https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/blochsphere/blochsphere.html

Use this relationship for (C) and (D): If $|q\rangle = a|0\rangle + b|1\rangle$, then $|q\rangle = \frac{a+b}{\sqrt{2}}|+\rangle + \frac{a-b}{\sqrt{2}}|-\rangle$.

(C) When measured in the $\{|+\rangle, |-\rangle\}$ basis, what is the probability that $|q\rangle$ will collapse into the state $|+\rangle$? (Find numerical answers by replacing a and b with the given values.)

(D) When measured in the $\{|+\rangle, |-\rangle\}$ basis, what is the probability that $|q\rangle$ will collapse into the state $|-\rangle$?

Use this relationship for (E) and (F): If $|q\rangle = a|0\rangle + b|1\rangle$, then $|q\rangle = \frac{a-ib}{\sqrt{2}}|+i\rangle + \frac{a+ib}{\sqrt{2}}|-i\rangle$.

(E) When measured in the $\{|+i\rangle, |-i\rangle\}$ basis, what is the probability that $|q\rangle$ will collapse into the state $|+i\rangle$?

(F) When measured in the $\{|+i\rangle, |-i\rangle\}$ basis, what is the probability that $|q\rangle$ will collapse into the state $|-i\rangle$?

9. Now consider the qubit when written in the form $|q\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$.

(A) Calculate again the answer to part (C) of question 8, substituting with $a = \cos\frac{\theta}{2}$ and $b = \sin\frac{\theta}{2}e^{i\phi}$ instead of their numerical values.

(B) Use $\theta = \frac{3\pi}{10}$ and $\phi = \frac{7\pi}{12}$ to calculate numerical answers using your formula from part (A). Does your answer match with your answer to 8(C)?

You may find these identities helpful:

$$\frac{e^{i\phi} + e^{-i\phi}}{2} = \cos\phi$$

$$2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta$$

Section 1: Measurements

You should be able to:

- *Identify the outcomes of measurement when presented with a generic state $|q\rangle$ on the Bloch sphere with $\{|0\rangle, |1\rangle\}$ being the measurement basis*
- *Calculate the probabilities of the measurement outcomes for $|q\rangle$ still with $\{|0\rangle, |1\rangle\}$ being the measurement basis, using both Dirac notation (expansion coefficients of the state written as a superposition of eigenstates of an operator corresponding to the observable being measured) and the representation on the Bloch sphere ($cos^2 \frac{\theta}{2}$ and $sin^2 \frac{\theta}{2}$)*
- *Describe the measurement outcomes and calculate the probabilities of those outcomes for the states $|0\rangle, |1\rangle, |+\rangle, |-\rangle, |+i\rangle, |-i\rangle$ when $\{|0\rangle, |1\rangle\}, \{|+\rangle, |-\rangle\}, \{|+i\rangle, |-i\rangle\}$ are each used as measurement bases, using a method of changing basis*
- *Describe the measurement outcomes and calculate the probabilities of those outcomes for the given generic state $|q\rangle$ when $\{|0\rangle, |1\rangle\}, \{|+\rangle, |-\rangle\}, \{|+i\rangle, |-i\rangle\}$ are each used as measurement bases (by changing basis)*
- *Describe that any vector $|q\rangle$ on the Bloch sphere can be used with the vector that resides on the opposite pole of the Bloch sphere $|-q\rangle$ as an orthonormal basis*
- *Describe the outcomes of measurement and the probabilities of measuring those outcomes when presented with the angle between an arbitrary state $|p\rangle$ and arbitrary measurement basis $\{|q\rangle, |-q\rangle\}$ on the Bloch sphere*

Orientation

Measurements as seen on the Bloch sphere

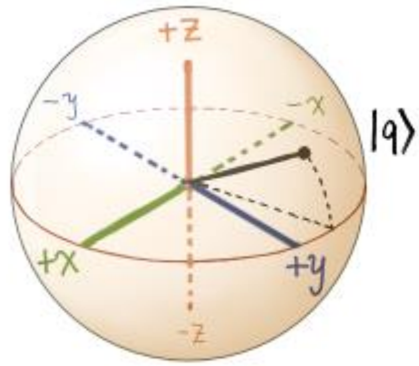
When a qubit is measured, the Bloch sphere offers a straightforward geometric interpretation regarding the probability of each possible outcome. First, a measurement basis is specified as usual, and the two basis states are located on the Bloch sphere: they will always be directly opposite each other. Then, once the state to be measured is plotted on the Bloch sphere, generally speaking, the basis state to which it leans closer is the state into which it has a higher probability of collapsing when measured.

10. Consider a qubit in the state $|q\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$. All measurements are made in the $\{|0\rangle, |1\rangle\}$ basis. Express your answers to (A) and (B) in terms of θ and ϕ.

(A) When measured in the $\{|0\rangle, |1\rangle\}$ basis, what is the probability that $|q\rangle$ will collapse into the state $|0\rangle$?

(B) What is the probability the measurement in part (A) will collapse $|q\rangle$ into the state $|1\rangle$?

(C) A state $|q\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$ is drawn on the Bloch sphere below. Label the angles θ and ϕ.

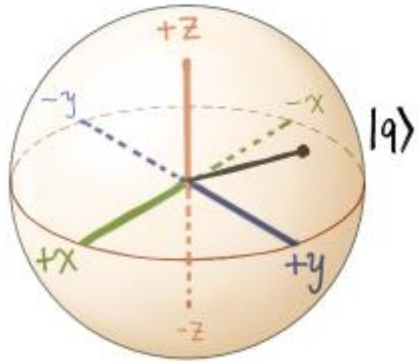


Use the simulation below to answer questions 10(D) and 10(E). Use $\theta = \frac{3\pi}{10}$ and $\phi = \frac{7\pi}{12}$ to represent the state $|q\rangle$. Click on the "Show theoretical measurement outcome probabilities" box to see them.
https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/blochsphere/blochsphere.html

(D) Starting in this state, change the angle ϕ while holding the angle θ fixed. Describe qualitatively what happens to the state as ϕ changes. How are the probabilities of measuring each outcome affected when a measurement in the $\{|0\rangle, |1\rangle\}$ basis is made?

(E) Starting in this state, change the angle θ while holding the angle ϕ fixed. Describe qualitatively what happens to the state as θ changes. How are the probabilities of measuring each outcome affected when a measurement in the $\{|0\rangle, |1\rangle\}$ basis is made?

(F) Again the state is drawn on the Bloch sphere, but this time the polar angle θ' is determined by starting from the $|1\rangle$ state (i.e., starting from the negative z-axis). Label θ' and express your answers to the following questions in terms of θ'.



(G) When measured in the $\{|0\rangle, |1\rangle\}$ basis, what is the probability that $|q\rangle$ will collapse into the state $|1\rangle$?

(H) What is the probability the measurement in part (G) will collapse $|q\rangle$ into the state $|0\rangle$?

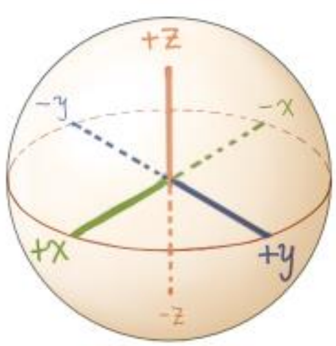
(I) Compare your answers to parts (A) and (H), and your answers to parts (B) and (G). Are your answers the same or different, and why?

Hint: Remember that $\sin(\frac{\pi}{2} - \psi) = \cos\psi$ and $\cos(\frac{\pi}{2} - \psi) = \sin\psi$.

Checkpoint
• On the Bloch sphere, when θ is the angle between the given state and a "target" basis state (not necessarily the closest basis state), the probability that the measurement will yield the target state is $\cos^2\frac{\theta}{2}$. The probability that the measurement will yield the state opposite the target state is $\sin^2\frac{\theta}{2}$. • For instance, if the state $\|q\rangle$ is measured in the $\{\|0\rangle, \|1\rangle\}$ basis with the target state considered to be $\|0\rangle$, then θ is the angle between $\|q\rangle$ and $\|0\rangle$. The probability that the state will collapse into $\|0\rangle$ is $\cos^2\frac{\theta}{2}$ and the probability that the state will collapse into $\|1\rangle$ is $\sin^2\frac{\theta}{2}$. • If the target state is $\|1\rangle$, and θ' is the angle between $\|q\rangle$ and $\|1\rangle$, then the probability that the state will collapse into $\|1\rangle$ is $\cos^2\frac{\theta'}{2}$, and the probability that the state will collapse into $\|0\rangle$ is $\sin^2\frac{\theta'}{2}$.

For question 11, you may find the following relations helpful, and a Bloch sphere is provided for reference (you can use your answers to question 4 to figure out the placement of the six basis states in the left column):

Starting basis		
$\{\|0\rangle, \|1\rangle\}$	$\|+\rangle = \frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$ $\|-\rangle = \frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$	$\|+i\rangle = \frac{1}{\sqrt{2}}(\|0\rangle + i\|1\rangle)$ $\|-i\rangle = \frac{1}{\sqrt{2}}(\|0\rangle - i\|1\rangle)$
$\{\|+\rangle, \|-\rangle\}$	$\|0\rangle = \frac{1}{\sqrt{2}}(\|+\rangle + \|-\rangle)$ $\|1\rangle = \frac{1}{\sqrt{2}}(\|+\rangle - \|-\rangle)$	$\|+i\rangle = \frac{1}{\sqrt{2}}(\|+\rangle - i\|-\rangle)$ $\|-i\rangle = \frac{1}{\sqrt{2}}(\|+\rangle + i\|-\rangle)$
$\{\|+i\rangle, \|-i\rangle\}$	$\|0\rangle = \frac{1}{\sqrt{2}}(\|+i\rangle + \|-i\rangle)$ $\|1\rangle = \frac{i}{\sqrt{2}}(\|-i\rangle - \|+i\rangle)$	$\|+\rangle = \frac{1}{\sqrt{2}}(\|-i\rangle + i\|+i\rangle)$ $\|-\rangle = \frac{1}{\sqrt{2}}(\|+i\rangle - i\|-i\rangle)$



11. What are the possible outcomes, and the probabilities of measuring those outcomes, when a qubit in one of six possible states $|0\rangle, |1\rangle, |+\rangle, |-\rangle, |+i\rangle, |-i\rangle$ is measured in each of the following bases? Show or explain your work. You may choose any method to calculate the probabilities.

(A) $\{|0\rangle, |1\rangle\}$ basis

State of the qubit	Possible outcomes	Probability of measuring each outcome	Is the qubit state parallel or perpendicular to the basis states (on the Bloch sphere)?
$\vert 0\rangle$			
$\vert +i\rangle$			

(B) $\{|+\rangle, |-\rangle\}$ basis

State of the qubit	Possible outcomes	Probability of measuring each outcome	Is the qubit state parallel or perpendicular to the basis states (on the Bloch sphere)?
$\vert 1\rangle$			
$\vert -\rangle$			

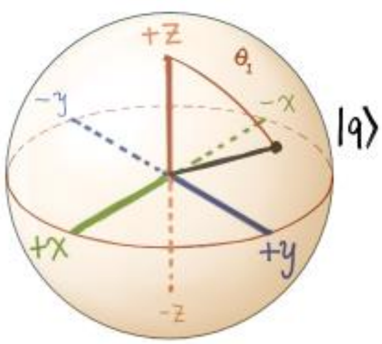
(C) $\{|+i\rangle, |-i\rangle\}$ basis

State of the qubit	Possible outcomes	Probability of measuring each outcome	Is the qubit state parallel or perpendicular to the basis states (on the Bloch sphere)?
$\vert +\rangle$			
$\vert -i\rangle$			

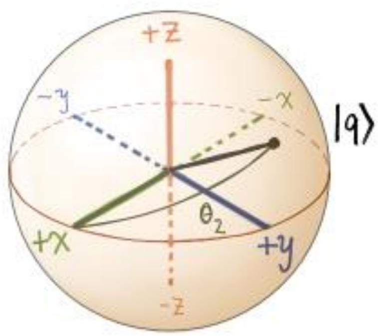
Checkpoint
• Measurements made in a given state that lies along one of the measurement basis states will yield that basis state with 100% probability. • Measurements made in a given state that lies perpendicular to the measurement basis states will yield either outcome with 50% probability.

For questions 12-13, consider a qubit in the state $|q\rangle$ illustrated in the following three diagrams, each one highlighting a different measurement basis.

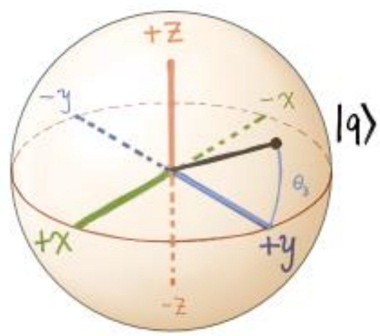
- In diagram (1), $|q\rangle$ makes an angle θ_1 with the $|0\rangle$ state, which lies along the $+z$ axis.
- In diagram (2), $|q\rangle$ makes an angle θ_2 with the $|+\rangle$ state (the $+x$ axis).
- In diagram (3), $|q\rangle$ makes an angle θ_3 with the $|+i\rangle$ state (the $+y$ axis).



(1)



(2)



(3)

12. Answer parts (A) through (F) in terms of θ_1, θ_2, and θ_3.

(A) When $|q\rangle$ is measured in the $\{|0\rangle, |1\rangle\}$ basis, what is the probability that $|q\rangle$ will collapse into the state $|0\rangle$?

(B) When $|q\rangle$ is measured in the $\{|0\rangle, |1\rangle\}$ basis, what is the probability that $|q\rangle$ will collapse into the state $|1\rangle$?

(C) When $|q\rangle$ is measured in the $\{|+\rangle, |-\rangle\}$ basis, what is the probability that $|q\rangle$ will collapse into the state $|+\rangle$?

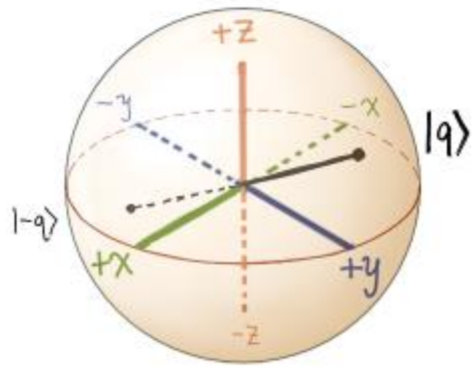
(D) When $|q\rangle$ is measured in the $\{|+\rangle, |-\rangle\}$ basis, what is the probability that $|q\rangle$ will collapse into the state $|-\rangle$?

(E) When $|q\rangle$ is measured in the $\{|+i\rangle, |-i\rangle\}$ basis, what is the probability that $|q\rangle$ will collapse into the state $|+i\rangle$?

(F) When $|q\rangle$ is measured in the $\{|+i\rangle, |-i\rangle\}$ basis, what is the probability that $|q\rangle$ will collapse into the state $|-i\rangle$?

If you did the warm-up, compare your answers to question 8 [warm-up] with what you just found for the last few diagrams: $\theta_1 = \frac{3\pi}{10}$ radians, $\theta_2 = 1.782$ radians and $\theta_3 = 0.674$ radians. Do your answers to questions 8 and 12 match?

13. Any two states that are found on opposite ends of a diameter of the Bloch sphere can be used as a measurement basis. Let us use the state $|q\rangle$ and its opposite state, $|-q\rangle$, as the measurement basis (illustrated below) for this question.



Answer parts (A) through (F) in terms of θ_1, θ_2, and θ_3 as defined in question 12.

(A) When the state $|0\rangle$ is measured in the $\{|q\rangle, |-q\rangle\}$ basis, what are the respective probabilities with which the measurement will yield the state $|q\rangle$ and the state $|-q\rangle$?

(B) When $|1\rangle$ is measured in this basis, what are the respective probabilities with which the measurement will yield the state $|q\rangle$ and the state $|-q\rangle$?

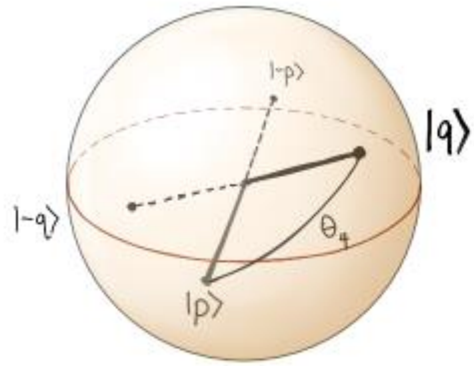
(C) When $|+\rangle$ is measured in this basis, what are the respective probabilities with which the measurement will yield the state $|q\rangle$ and the state $|-q\rangle$?

(D) When $|-\rangle$ is measured in this basis, what are the respective probabilities with which the measurement will yield the state $|q\rangle$ and the state $|-q\rangle$?

(E) When $|+i\rangle$ is measured in this basis, what are the respective probabilities with which the measurement will yield the state $|q\rangle$ and the state $|-q\rangle$?

(F) When $|-i\rangle$ is measured in this basis, what are the respective probabilities with which the measurement will yield the state $|q\rangle$ and the state $|-q\rangle$?

14. Now also consider the state $|p\rangle$ and its orthogonal state (the state found opposite to it), $|-p\rangle$. The angle between $|p\rangle$ and $|q\rangle$ is θ_4.



(A) When $|q\rangle$ is measured in the $\{|p\rangle, |-p\rangle\}$ basis, what are the outcomes of the measurement, and what are the respective probabilities of measuring those outcomes?

(B) When $|p\rangle$ is measured in the $\{|q\rangle, |-q\rangle\}$ basis, what are the outcomes of the measurement, and what are the respective probabilities of measuring those outcomes?

(C) When $|-p\rangle$ is measured in the $\{|q\rangle, |-q\rangle\}$ basis, what are the outcomes of the measurement, and what are the respective probabilities of measuring those outcomes?

15. Consider your answers to questions 12-14. What are the cases in which you found the same probabilities? Can you find a pattern?

Checkpoint
• When calculating the probability of a given state collapsing into one of the measurement basis states, the only important considerations are the angle between the given state and the states of the measurement basis. • When θ is the angle between a given state and one of the measurement basis states, the probability of the measurement yielding that measurement basis state is $\cos^2\frac{\theta}{2}$, and the probability of it yielding the opposite measurement basis state is $\sin^2\frac{\theta}{2} = \cos^2\frac{\pi-\theta}{2}$. • This means that you can always use $\cos^2\frac{\theta}{2}$ to determine the probability with which a measurement will yield a target state in the measurement basis, provided that θ is the angle between the given state and the target state (not necessarily the angle between the given state and the nearest measurement basis state). • Any two states that are found on opposite ends of the Bloch sphere can be used as a measurement basis. This means that any direction can be chosen as the z-axis! • Once a measurement is performed in a qubit state, it is more likely to collapse toward the basis state toward which it "leans closer."

Section 2: Geometric intuition

You should be able to:

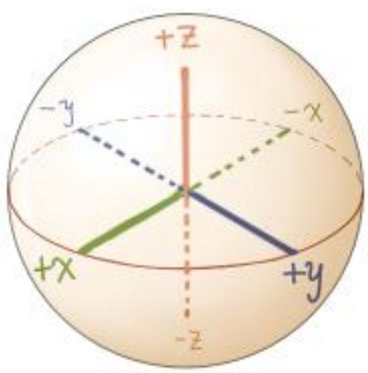
- *Identify states for which measurements in a particular basis yield a result with 100% probability or 100% certainty (i.e., eigenstates of the measurement basis)*
- *Identify that a circle represents the set of all states that a single measurement basis can distinguish from all other states, since the measurement basis cannot determine relative phase in that basis*
- *Describe that no two distinct measurement bases on the Bloch sphere are compatible; an eigenstate of one basis cannot be an eigenstate of any other basis*
- *Describe that, in general, a minimum of three measurement bases are needed to completely determine the state of a qubit*

Orientation

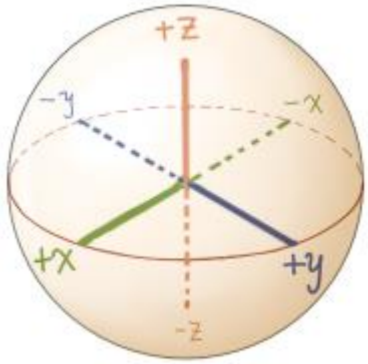
The Bloch sphere is especially useful for thinking about qubit behavior visually. Certain situations are very nice to think about using the Bloch sphere.

If a measurement will yield an outcome with 100% probability, then the measurement is said to be *certain*. This is in contrast to a measurement that might yield either outcome with nonzero probability, which implies some level of *uncertainty* in the measurement.

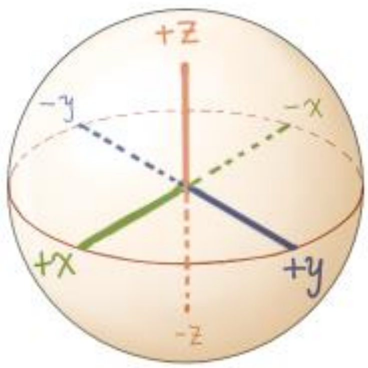
16. In the $\{|0\rangle, |1\rangle\}$ measurement basis, what are all the states for which a measurement outcome is certain (i.e., the outcome will be measured with 100% probability)? Identify them on the Bloch sphere below.



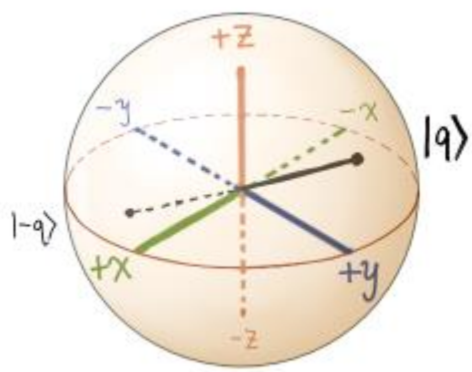
17. In the $\{|0\rangle, |1\rangle\}$ basis, what is the set of all states that will yield each of the basis states with 50% probability? (Hint: $|+\rangle, |-\rangle, |+i\rangle$, and $|-i\rangle$ are not the only such states.) Identify the states on the Bloch sphere below.



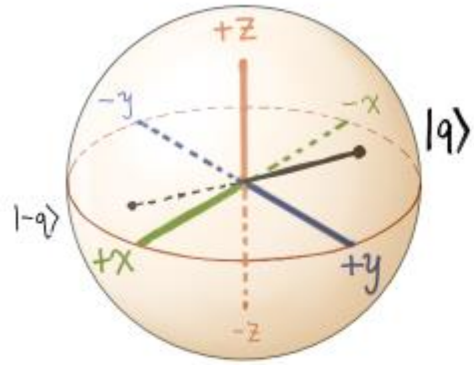
18. In the $\{|0\rangle, |1\rangle\}$ basis, what is the set of all states that will yield the $|0\rangle$ state with 70% probability and the $|1\rangle$ state with 30% probability? Identify the states on the Bloch sphere below. (Make a rough approximation.)



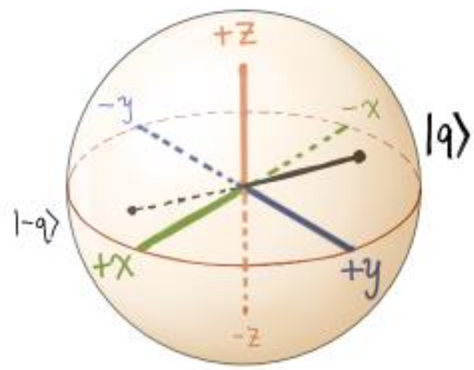
19. In the $\{|q\rangle, |-q\rangle\}$ basis, what are all the states for which the measurement outcome is certain? Identify the states on the Bloch sphere below.



20. In the $\{|q\rangle, |-q\rangle\}$ basis, what is the set of all states that will yield each of the basis states with 50% probability? Identify the states on the Bloch sphere below.

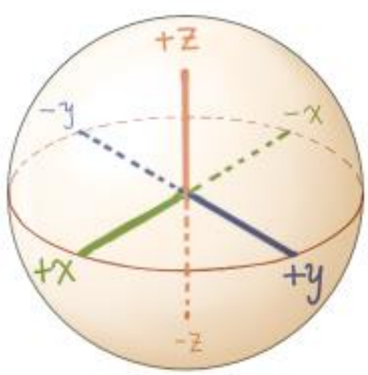


21. In the $\{|q\rangle, |-q\rangle\}$ basis, what is the set of all states that will yield the $|q\rangle$ state with 90% probability and the $|-q\rangle$ state with 10% probability? Identify the states on the Bloch sphere below. (Make a rough approximation.)

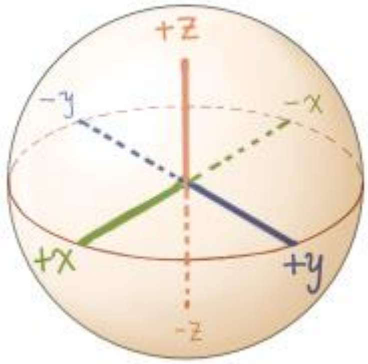


22. When a measurement is made in either the $\{|0\rangle, |1\rangle\}$ or $\{|+\rangle, |-\rangle\}$ basis, does there exist some state for which both measurements are certain (i.e. yield some outcome with 100% probability)? We will try to reach an answer in the following parts to this question.

(A) As in question 16, draw on the Bloch sphere below all of the states for which a measurement outcome is certain in the $\{|0\rangle, |1\rangle\}$ basis.



(B) Draw on the Bloch sphere below all of the states for which a measurement outcome is certain in the $\{|+\rangle, |-\rangle\}$ basis.

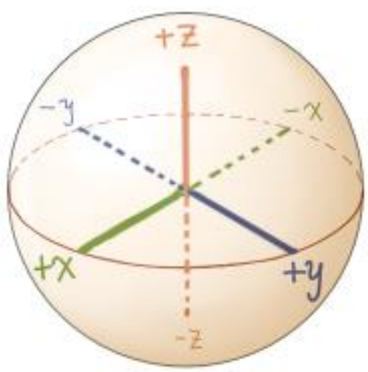


(C) Are there any states that are members of both sets?

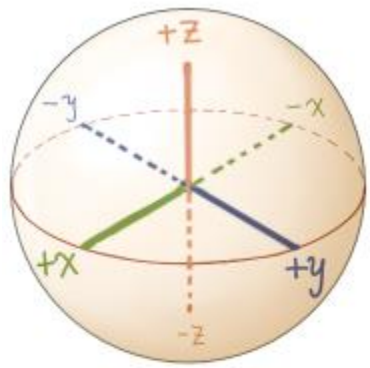
(D) Generalize this to other bases. For *any* two different bases, is it possible to find a state that yields a 100% certain outcome when a measurement is made in each basis (on identically-prepared qubits—not in succession)?

23. Instead of 100% certainty, is it possible for a measurement to yield either outcome with 50% probability in multiple bases at once? We will find out in the following parts.

(A) On the Bloch sphere below, first draw the set of states that, when measured in the $\{|0\rangle, |1\rangle\}$ basis, yields each outcome with 50% probability. Then draw the set of states that yields each outcome with 50% probability in the $\{|+\rangle, |-\rangle\}$ basis. Indicate clearly any points of overlap.



(B) On the Bloch sphere below, first draw the set of states that, when measured in the $\{|+\rangle, |-\rangle\}$ basis, yields each outcome with 50% probability. Then draw the set of states that yields each outcome with 50% probability in the $\{|+i\rangle, |-i\rangle\}$ basis. Indicate clearly any points of overlap.



(C) Does there exist some state on the Bloch sphere that, when measured in all three of the $\{|0\rangle, |1\rangle\}$, $\{|+\rangle, |-\rangle\}$, and $\{|+i\rangle, |-i\rangle\}$ bases (which are all independent), yields each outcome with 50% probability in each basis?

(To answer this question, look at your two diagrams for the previous two problems. Are there any points where all three sets of states should overlap? I.e., are there points of overlap that are shared between the two diagrams?)

Note: This tutorial only discusses pure states, which are represented by points on the surface of the Bloch sphere. The answer may be different if one starts with a mixed state—but we will not deal with mixed states.